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On a generalized Uncertainty Principle, coherent states, and the moment map

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It is shown that, under general circumstances, symplectic G-orbits in a hamiltonian manifold acted on (symplectically) by a Lie group G provide critical points for the norm squared of the moment map. This fact yields a "variational" interpretation of the symplectic orbits appearing in the projective space attached to an irreducible representation of a compact simple Lie group (according to work of Kostant and Sternberg and of Giavarini and Onofri), where the previous function is also related to the invariant uncertainty considered by Delbourgo and Perelomov.

A notion of generalized canonical conjugate variables (in the Kähler case) is also presented and used in the framework of a Kähler geometric interpretation of the Heisenberg uncertainty relations (building on the analysis given by Cirelli, Manià and Pizzocchero and by Provost and Vallee); it is proved, in particular, that the generalized coherent states of Rawnsley minimize the uncertainty relations for any pair of generalized canonically conjugate variables.

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1. Introduction

Let (M, Ω) be a symplectic manifold and let $\mu: M \to \mathfrak{g}^*$ be a moment map associated to a symplectic action on M of a compact, simple, simply connected Lie group G with Lie algebra $\mathfrak{g}$ (which we identify with $\mathfrak{g}^*$ by means of a G-invariant metric inducing a norm denoted by $\| \; \|$). The function $\|\mu\|^2$ has then remarkable properties as a minimally degenerate Morse function [19], deeply explored in a number of interesting papers (e.g. refs. [1,14,18,19,24]), where the structure of its critical points and its cohomological implications in geometric invariant theory have been uncovered. The prototype of such a function is the Yang–Mills functional (cf., e.g. ref. [3]). In this paper we investigate further the properties of this function.

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First, we prove that under very general conditions, any *symplectic* orbit of G in M is always a *critical* orbit for $\|\mu\|^2$ (theorem 2.1; the converse is, however, false). We apply this remark to the projective space $P(V)$ determined by an irreducible representation of a compact simple Lie group on a hermitian space V, rendering some results of Kirwan and Ness a little bit more explicit in this particular case (theorem 5.3.).

A possibly interesting observation is that $\|\mu\|^2$, in this particular case, is essentially (up to a negative scalar and an additive constant) the *invariant dispersion* first introduced by Delbourgo in ref. [11] (and generalized to the present context in, e.g., ref. [26]). We naturally recover the result that the highest weight (Kähler) orbit (made up of coherent states) minimizes the invariant dispersion [11,26], and at the same time provides a natural "variational" interpretation for the appearance of the other symplectic G-orbits in projective space (theorem 5.3 again).

However, the symplectic G-orbits play an important role in the interpretation of the phenomenon of spin *migration* in the context of the degeneracy of Landau levels [12], so our results support a Morse theoretic interpretation of the above phenomenon.

We further pursue the uncertainty problem for coherent states in the framework of geometric quantization. It is well known (see, e.g., ref. [26]) that the ordinary coherent states possess a number of remarkable physical properties; one of the most characteristic is that they provide minimum uncertainty wave packets with respect to the position operator and its canonically conjugate momentum operator (referred to any particular variable) and, moreover, the dispersions of these observables are equal.

We analyze the persistence of this property in the context of the generalized coherent states on a quantizable Kähler manifold introduced by Rawnsley in ref. [28], and further studied in ref. [6], with respect to a suitably generalized concept of canonically conjugate quantum variables. This concept turns out to be *state dependent* (section 3). This is done by resorting to the geometrical interpretation of the dispersion structure of ordinary quantum mechanics set up in refs. [10,27], which we recall in section 3. We show, roughly speaking, that the minimum uncertainty property still holds in this broader framework (theorem 3.5) and we again examine explicitly the Kähler orbit case (theorem 4.2).

2. Preliminaries: symplectic geometry and geometric quantization

In this section we shall mostly refer to refs. [14–16,19,20,28,6]. It is mainly intended to fix notation and ensure readability.

Let (M, Ω, G) be a symplectic hamiltonian manifold acted on by a Lie group G (taken from the outset to be compact, simple, connected and simply connected), with Lie algebra $\mathfrak{g}$, and *hamiltonian algebra* Λ and corresponding G-equivariant *moment map* $\mu: M \to \mathfrak{g}^*$ given by (for $m \in M$, $X \in \mathfrak{g}$)

$$\langle \mu(m), X \rangle = \lambda_X(m) , \tag{2.1}$$

with $\lambda_X \in \Lambda \subset C^\infty(M)$ and with $\langle \, , \, \rangle$ denoting the duality pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. Further, $X^\#$ denotes the fundamental vector field on M induced by $X \in \mathfrak{g}$ and we have

$$\Omega(X^\#, \cdot) = \mathrm{d}\lambda_X . \tag{2.2}$$

The *Poisson bracket* $\{ \, , \, \}$ on $C^\infty(M)$ is defined through the symplectic form in the standard way,

$$\{f, g\} := \Omega(f^\#, g^\#) , \tag{2.3}$$

where $f^\#$ denotes the smooth vector field on M induced by f via (2.2) (using the non-degeneracy of Ω).

We shall always identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by means of a G-invariant metric $\| \; \|^2$. In particular, we may think of the moment map as taking values in $\mathfrak{g}$, and consider the G-invariant function on M given by $\|\mu\|^2$. The remarkable properties of this function are well known (see, e.g., refs. [19,24,2]).

However, the following property does not seem to have been explicitly observed before. In order to state it, we first recall that, with the above notation, $x \in M$ is critical for $\|\mu\|^2$ if and only if (see, e.g., ref. [19])

$$\mu(x)^\# |_x = 0 . \tag{2.4}$$

Theorem 2.1. *Let (M, Ω, G) be as above. Let $x \in M$. If the G-orbit $G \cdot x$ is symplectic, then x (and hence any point of the orbit) is a critical point for $\|\mu\|^2$.*

Proof. It is obvious that $\mu(x) \in \mathfrak{g}$ belongs to $\mathfrak{g}_{\mu(x)}$. Let $\xi \in \mathfrak{g}_{\mu(x)}$. Then we have, for any $\eta \in \mathfrak{g}$,

$$\begin{aligned} \Omega_x(\xi^\#|_x, \eta^\#|_x) &= \lambda_{[\xi, \eta]}(x) = \langle \mu(x), [\xi, \eta] \rangle \\ &= K_{\mu(x)}(\xi^\#|_{\mu(x)}, \eta^\#|_{\mu(x)}) = 0 , \end{aligned} \tag{2.5}$$

which, by the non-degeneracy of Ω when restricted to $G \cdot x$, yields $\xi^\#|_x = 0$. This is valid, in particular, for $\mu(x)$. Here K denotes the Kirillov–Kostant–Souriau symplectic structure on the G-coadjoint orbit through $\mu(x)$. The same conclusion could have been reached using theorem 26.8 of ref. [16]. □

The previous result may be used to detect critical points of $\|\mu\|^2$ in specific examples.

We shall be acting in the framework of geometric quantization of Kähler manifolds and their coherent states, referring to refs. [6,28] for a thorough treatment.

Let J, here and henceforth, denote the *complex structure* of the Kähler manifolds involved. Let V be a complex Hilbert space, equipped with a hermitian scalar product $\langle\ ,\ \rangle$ linear in the second variable. We freely employ Dirac's bra–ket notation. Let $P(V)$ be its associated projective space, consisting of the one-dimensional complex vector subspaces of V. If v is a non-zero vector, we denote the one-dimensional complex vector space it generates by $[v]$. A point $[v]$ in $P(V)$ can also be effectively represented by the orthogonal projection operator onto $[v]$, namely $\|v\|^{-2}|v\rangle\langle v|$. Points in $P(V)$ represent, physically, the *quantum states* of a specific physical system. The mean expectation value of a *quantum observable* (namely, a linear self-adjoint operator A on H) in the state $[v]$ (provided v is in the domain of A) is given by

$$[v](A) \equiv \langle A\rangle_{[v]} := \|v\|^{-2}\langle v|Av\rangle\ , \qquad (2.6)$$

and can naturally be interpreted in terms of the moment map for $P(V)$ (see below). We shall often use the shorthand notation $\langle A\rangle$ as well.

Let (M, Ω) be a finite-dimensional Kähler manifold, with Ω *integral* and *positive*, and let $L \to M$ be a hermitian, holomorphic, *regular* [28,6] line bundle obtained via the geometric quantization procedure (see, e.g., refs. [20,15,16,28,6,13]), which realizes the *coherent state* embedding $\varepsilon: M \to P(V)$, V being the quantum Hilbert space, consisting of the square integrable holomorphic sections of L. We shall assume that $\varepsilon^*\Omega_F = \Omega$, with Ω_F denoting the canonical Kähler (Fubini–Study) form on $P(V)$ (see below for its explicit expression). This condition holds in most interesting examples [7]. In this paper we shall mainly deal with a *finite-dimensional* V, although part of our discussion easily extends to the infinite-dimensional case.

For later use we recall that another way of expressing holomorphy is via the notion of *complex polarization* (see, e.g., ref. [15] for details): $\Gamma(F)$ will denote the module of sections of the antiholomorphic polarization F. Recall that, if λ is a *quantizable* [6] *classical* observable, namely, λ is a smooth function on M such that $\lambda^\#$ preserves the anti-holomorphic polarization F (i.e., $[\lambda^\#, \Gamma(F)] \subseteq \Gamma(F)$), then it gives rise to a *quantum* observable Q_λ on V.

Proposition 2.2 (cf. ref. [6]). *Let λ be a quantizable hamiltonian and let Q_λ be its corresponding quantum observable. Then $\varepsilon^*\langle Q_\lambda\rangle = \lambda$, i.e.,*

$$\lambda(x) = \langle Q_\lambda\rangle(\varepsilon(x))\ . \quad \square \qquad (2.7)$$

This result expresses the fact that the expectation value of a quantum observable on a coherent state equals the value of the classical observable at the point involved, so that it undergoes a *classical* evolution. We have used a slightly different notation from ref. [6].

We now briefly review the relevant differential geometry of $P(V)$. We mostly refer to ref. [16]. It may also be worthwhile to observe that we shall recover the basic results of ref. [27] within our formalism. An advantage of the present formalism is that it is completely intrinsic.

Let V' denote the orthogonal complement to the line in $[v]$ in V $(v\neq 0)$ with respect to $\langle\ |\ \rangle$. We shall need the following lemma, which is established through an immediate calculation.

Lemma 2.3. *Let* $v, w\in V$, $\|v\|=1$, $w:=w_1+w_2$, *with* $w_1\in[v]$, $w_2\in V'$. *Set* $v(t):=v+tw$. *Then*

$$(\mathrm{d}/\mathrm{d}t)\{\langle v(t)|v(t)\rangle^{-1}|v(t)\rangle\langle v(t)|\}|_{t=0}$$
$$=|w_2\rangle\langle v|+|v\rangle\langle w_2|\ . \qquad \square \tag{2.8}$$

This formula says, in particular, that the tangent space $T_{[v]}(P(V))$ can be identified with V'.

The projective space $P(V)$ is a $U(V)$-homogeneous Kähler manifold [with $U(V)$ denoting the unitary group of V]. The Lie algebra of $U(V)$, consisting of all skew-hermitian matrices, will be denoted by $\mathrm{u}(V)$. Due to homogeneity, the tangent space at any point of $P(V)$ is spanned by the fundamental vector fields (evaluated at the point in question) induced by elements in $\mathrm{u}(V)$. The *fundamental vector field* $A^{\#}$ attached to $A\in\mathrm{u}(V)$ reads, at $[v]\in P(V)$ $(\|v\|=1)$,

$$A^{\#}|_{[v]}=|v\rangle\langle Av|+|Av\rangle\langle v|\ . \tag{2.9}$$

The *complex structure* J reads, at $[v]\in P(V)$ $(\|v\|=1)$,

$$J|_{[v]}A^{\#}|_{[v]}=|v\rangle\langle \mathrm{i}Av|+|\mathrm{i}Av\rangle\langle v|\ . \tag{2.10}$$

One immediately gets the formula for the Kähler metric g_{F} of $P(V)$ and Fubini–Study form Ω_{F} (cf. refs. [16,9,27]) (where A, B are skew-hermitian, $\|v\|=1$ and we recall that $\mathrm{Tr}(|v\rangle\langle w|)=\langle w|v\rangle$),

$$g_{F[v]}(A^{\#}|_{[v]},B^{\#}|_{[v]})=\mathrm{Re}\{\langle Av|Bv\rangle+\langle v|Av\rangle\langle v|Bv\rangle\}\ ,$$
$$\Omega_{F[v]}(A^{\#}|_{[v]},B^{\#}|_{[v]})=g_{F[v]}(J|_{[v]}A^{\#}|_{[v]},B^{\#}|_{[v]})$$
$$=\mathrm{i}/2\langle v|[A,B]|v\rangle\ . \tag{2.11}$$

The canonical inner product in $\mathrm{u}(V)$ is given by $(A,B):=-\frac{1}{2}\mathrm{Tr}(AB)$. The *moment map* reads (cf. refs. [2,19,24])

$$\mu([v])=-\mathrm{i}|v\rangle\langle v|\in\mathrm{u}(V)\ , \tag{2.12}$$

i.e., if $A\in\mathrm{u}(V)$,

$$\mu_A([v])=(\mu([v]),A)=-\tfrac{1}{2}\mathrm{Tr}(-\mathrm{i}|v\rangle\langle v|A)=\tfrac{1}{2}\mathrm{i}\langle v|A|v\rangle\ .$$

One can easily deduce, by means of the preceding formulae, the following "compatibility" result, which will be needed in the sequel.

Theorem 2.4. *Let us assume that we are given a unitary action of a compact Lie group G (with Lie algebra* $\mathfrak{g}$*) on V and hence on P(V) (thus* $\mathfrak{g}\subseteq\mathfrak{u}(V)$*). Then*

$$\mu^{\#}_{A}|_{O}=A^{\#}|_{O}, \tag{2.13}$$

for any $A\in\mathfrak{g}$*, and any* symplectic *G-orbit O in P(V).* □

This result applies, in particular to the unique Kähler G-orbit in $P(V)$ (see section 4).

3. A generalized Uncertainty Principle

Let us briefly recall the *dispersion structure* of ordinary quantum mechanics [10,27]. Let

$$\Delta_{[v]}(B):=\Delta^{2}_{[v]}(B)^{1/2}=(\langle B^{2}\rangle_{[v]}-\langle B\rangle^{2}_{[v]})^{1/2}$$

be, as usual, the dispersion of the quantum observable B in the state $[v]$. We have the following

Proposition 3.1.

(i) Let A and B be any two quantum observables and let $[v]\in P(V)$*. Then (*Uncertainty Relations*)*

$$\Delta_{[v]}(A)\Delta_{[v]}(B)\geqslant\tfrac{1}{2}|\langle[A,B]\rangle_{[v]}|\,. \tag{3.1}$$

(ii) (cf. ref. [10]). Let B be a quantum observable. Then

$$\Delta_{[v]}(B)=g_{F[v]}(\mu^{\#}_{\mathrm{i}B}|_{[v]},\mu^{\#}_{\mathrm{i}B}|_{[v]})^{1/2}\,. \tag{3.2}$$

Proof. (i) follows immediately from (2.11) and from the Schwarz inequality. (ii) also follows from (2.11) and theorem 2.4:

$$g_{Fx}(\mu^{\#}_{\mathrm{i}B}|_{[v]},\mu^{\#}_{\mathrm{i}B}|_{[v]})=g_{F[v]}(\mathrm{i}B^{\#}|_{[v]},\mathrm{i}B^{\#}|_{[v]})=\langle B^{2}\rangle_{[v]}-\langle B\rangle^{2}_{[v]}$$

(for the last equality see ref. [27] as well). □

The previous result shows that, in a sense, complex geometry is "stored" in quantum mechanics in a natural way at a fundamental level (see ref. [10], and references therein). We have seen that, within this geometrical reinterpretation, the Uncertainty Relations appear as a restatement of *Schwarz's inequality* [this remains true within the C^{*}-algebraic approach to quantum mechanics as well (by the Gelfand–Naimark–Segal theorem)]. This elementary fact will be important for what follows.

Let (M, Ω) be a Kähler manifold, with complex structure J.

Definitions 3.2.

(i) Let λ be a hamiltonian (classical observable) and let $\lambda^{\#}$ be its corresponding hamiltonian vector field on M. A *canonically conjugate* classical observable to λ *at* $x \in M$ is a hamiltonian λ' such that its hamiltonian vector field $\lambda'^{\#}$ equals $J\lambda^{\#}$ *when evaluated at* x:

$$\lambda'^{\#}|_x = J|_x \lambda^{\#}|_x .$$

(ii) Similarly, $Q_{\lambda'}$ will be a quantum observable *canonically conjugate* to Q_λ at $\varepsilon(x)$, provided both exist.

Notice that, if λ' is canonically conjugate to λ, then λ is canonically conjugate to $-\lambda'$, as $J^2 = -I$. Moreover, λ' is not unique.

For the sake of brevity, we declare two hamiltonians to be canonically conjugate at x if their corresponding vector fields, at x, differ by the action of $\pm J$.

The geometrical reason for these definitions is again simply that the Schwarz inequality for the Kähler metric g corresponding to Ω reads

$$\begin{aligned} & g|_x(\lambda^{\#}|_x, \lambda^{\#}|_x)^{1/2} g|_x(\mu^{\#}|_x, \mu^{\#}|_x)^{1/2} \\ & = g|_x(\lambda^{\#}|_x, \lambda^{\#}|_x)^{1/2} g|_x(J|_x\mu^{\#}|_x, J|_x\mu^{\#}|_x)^{1/2} \\ & \geqslant |g|_x(\lambda^{\#}|_x, J|_x\mu^{\#}|_x)| , \end{aligned}$$

with equality holding (at x) if and only if $\lambda^{\#}|_x = J|_x\mu^{\#}|_x$, i.e., if λ is canonically conjugate to μ at x.

Observe that, at least locally, given a hamiltonian, one can construct a canonically conjugate hamiltonian to it. In fact, taking a non-characteristic hypersurface containing x for the linear partial differential equation (in the vector field X) $L_X\Omega = i_X\Omega = 0$, subject to the condition $X|_x = J|_x\lambda^{\#}|_x$, we see that it is locally solvable by standard methods. Otherwise one can employ the Darboux theorem, and further observe that at the origin x of the Darboux coordinates, the complex structure J is given by the standard complex structure in $\mathbb{C}^n$.

These definitions are physically justified by the fact that the position and momentum observables of ordinary Schrödinger mechanics indeed satisfy them (and are actually state independent). This is the content of proposition 3.4 below. The necessity of giving a state-dependent definition of canonically conjugate variables also stems from the fact that, if a vector field $\xi^{\#}$ is hamiltonian, this does *not* imply in general that $J\xi^{\#}$ is still hamiltonian (nor even symplectic). However, we shall see that many physically interesting observables are canonically conjugate with respect to our definition (proposition 3.4 and theorem 4.2 below). Nevertheless, the following is true.

Proposition 3.3. *Let X be a simply connected Kähler manifold. Let $\xi^{\#}$ be a hamiltonian vector field, with hamiltonian λ_ξ. Then if $J\xi^{\#}$ is hamiltonian (with hamiltonian $\lambda_{J\xi}$) then $\lambda_{J\xi}$ is quantizable.*

Proof. The proof follows by direct computation or by resorting to ref. [15], theorem 4.5. In fact, if $\xi^{\#}$ preserves F (i.e., $[\xi^{\#}, \Gamma(F)] \subseteq \Gamma(F)$, where $\Gamma(F)$ denotes the module of sections of F), then $J\xi^{\#}$ preserves F as well; so, provided it is hamiltonian, its hamiltonian is quantizable as well. □

Proposition 3.4. *Let $X = \mathbb{C}^n$, equipped with its standard symplectic Kähler form, and let it be quantized in the usual way (à la Bargmann–Fock, see, e.g., refs. [6,26]). Then the hamiltonians representing the position and momentum observables, respectively, are quantizable and canonically conjugate in the above sense.*

Sketch of proof. The vector fields attached to these observables are (up to signs) partial differentiation operators with respect to the canonical coordinates and, having constant coefficients, preserve F, and are clearly hamiltonian. This also follows from the discussion of this example given in ref. [6], part I, where all quantizable functions are explicitly determined. □

Again from ref. [6], we know that there may be only a few quantizable functions, in general (Groenwald–van Hove phenomenon, see, e.g., ref. [16] for a detailed discussion). Nevertheless, in the *compact* case and when the quantizations of L^k are *regular* (for all $k \geqslant 1$) there is a *dense* set of quantizable functions [6].

The following is the basic result of this section. Observe in particular that (ii) will yield a somewhat *intrinsic* characterization of the dispersion of a quantum observable in a coherent state in terms of the Kähler metric of the coherent state manifold, and provide a natural generalization of the basic observation of ref. [10] (see also ref. [27]).

Theorem 3.5.

(i) (coherence property)

$$\Omega|_x(\lambda_1^{\#}|_x, \lambda_2^{\#}|_x) = \Omega_F|_{\varepsilon(x)}(\mathrm{i}Q_{\lambda_1}^{\#}|_{\varepsilon(x)}, \mathrm{i}Q_{\lambda_2}^{\#}|_{\varepsilon(x)}) .$$

(ii)

$$\Delta_{\varepsilon(x)}(Q_\lambda) = g|_x(\lambda^{\#}|_x, \lambda^{\#}|_x)^{1/2} .$$

(iii) The dispersions of any two canonically conjugate quantum variables Q_λ and $Q_{\lambda'}$ (when they exist) in a generalized coherent state are equal and minimize the Uncertainty Relations.

Proof.

(i)
$$\begin{aligned}\Omega|_x(\lambda_1^\#|_x, \lambda_2^\#|_x) &= \{\lambda_1, \lambda_2\}(x) = \{\langle Q_{\lambda_1}\rangle, \langle Q_{\lambda_2}\rangle\}(\varepsilon(x)) \\ &= \Omega_F|_{\varepsilon(x)}(\langle \mathrm{i}Q_{\lambda_1}\rangle^\#|_{\varepsilon(x)}, \langle \mathrm{i}Q_{\lambda_2}\rangle^\#|_{\varepsilon(x)}) \\ &= \Omega_F|_{\varepsilon(x)}(\mathrm{i}Q_{\lambda_1}^\#|_{\varepsilon(x)}, \mathrm{i}Q_{\lambda_2}^\#|_{\varepsilon(x)}) \\ &= -\tfrac{1}{2}\langle [Q_{\lambda_1}, Q_{\lambda_2}]\rangle_{\varepsilon(x)} \\ &= \Omega_F|_{\varepsilon(x)}(\mathrm{i}Q_{\lambda_1}^\#|_{\varepsilon(x)}, \mathrm{i}Q_{\lambda_2}^\#|_{\varepsilon(x)}) .\end{aligned}$$

(ii) $\Delta_{\varepsilon(x)}(Q_\lambda)^2 = g_F|_{\varepsilon(x)}(\langle \mathrm{i}Q_\lambda\rangle^\#|_{\varepsilon(x)}, \langle \mathrm{i}Q_\lambda\rangle^\#|_{\varepsilon(x)}) = g|_x(\lambda^\#|_x, \lambda^\#|_x)$

(since $\varepsilon^* g_F = g$ and $\varepsilon^*\langle Q_\lambda\rangle = \lambda$).

(iii)
$$\begin{aligned}\Delta_{\varepsilon(x)}(Q_\lambda) &= g|_x(\lambda^\#|_x, \lambda^\#|_x)^{1/2} = g|_x(J|_x\lambda^\#|_x, J|_x\lambda^\#|_x)^{1/2} \\ &= g|_x(\lambda'^\#|_x, \lambda'^\#|_x)^{1/2} = \Delta_{\varepsilon(x)}(Q_{\lambda'}).\end{aligned}$$

Moreover

$$\begin{aligned}\Delta_{\varepsilon(x)}(Q_\lambda)\cdot\Delta_{\varepsilon(x)}(Q_{\lambda'}) &= g|_x(\lambda^\#|_x, \lambda^\#|_x) \\ &= \Omega|_x(\lambda^\#|_x, J|_x\lambda^\#|_x) = \Omega|_x(\lambda^\#|_x, \lambda'^\#|_x) \\ &= \Omega_F|_{\varepsilon(x)}(\langle \mathrm{i}Q_\lambda\rangle^\#|_{\varepsilon(x)}, \langle \mathrm{i}Q_{\lambda'}\rangle^\#|_{\varepsilon(x)}) \\ &= \tfrac{1}{2}|\langle [Q_\lambda, Q_{\lambda'}]\rangle_{\varepsilon(x)}| ,\end{aligned}$$

so the Uncertainty Relations are minimized. □

4. The projective space associated to an irreducible representation of a compact semisimple Lie group: the Kähler G-orbit

In this section we specialize the general discussion given in the preceding sections to the projective space associated to an irreducible representation space V of a compact, connected, simply connected, simple Lie group G.

We shall make free use of the basic notions and terminology pertaining to the theory of compact semisimple Lie groups, which is discussed in full detail in many excellent treatises (see, e.g., ref. [17]; see also refs. [26,16,4]). Let us denote by $\mathfrak{h}$ a fixed Cartan subalgebra of $\mathfrak{g}$ (the Lie algebra of a maximal torus H of G). The adjoint action of H on $\mathfrak{g}_{\mathbb{C}}$, the complexification of $\mathfrak{g}$, gives rise to an orthogonal decomposition (using the G-invariant scalar product induced by the Killing form B)

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \sum_\alpha \mathbb{C}E_\alpha \oplus \sum_\alpha \mathbb{C}E_{-\alpha} \tag{4.1}$$

(the sums ranging over the positive *roots*, with respect to a given positive *Weyl* *chamber*, in $\mathfrak{H} \cong \mathfrak{H}^*$, denoted by $\mathfrak{H}_+$). The roots can be interpreted as vectors in the real vector space $\mathfrak{H} \cong \mathfrak{H}^*$, which comes endowed with a natural G-invariant scalar product denoted by $\cdot$. Let R denote the set of all roots. Recall the following relations:

$$
\begin{aligned}
[E_\alpha, E_\beta] &= N_{\alpha,\beta} E_{\alpha+\beta}, && \text{if } \alpha+\beta\ (\neq 0) \text{ is a root}, \\
&= 0 && \text{otherwise}, \\
[E_\alpha, E_{-\alpha}] &= \mathrm{i} H_\alpha, \\
[H, E_\alpha] &= \mathrm{i}\alpha(H) E_\alpha \quad (H, H_\alpha \in \mathfrak{H}), && \qquad (4.2)
\end{aligned}
$$

where the $N_{\alpha,\beta}$ can be chosen to be integers and obey identities (which we shall not need) reflecting the Lie algebra relations. We notice here that, if λ is a *weight* for some representation of G, then $\lambda(\mathrm{i}H_\alpha) = \lambda \cdot \alpha$ (see, e.g., ref. [17], and below), where λ is interpreted as an element of $\mathfrak{H}^*$ in the l.h.s. and as an element of $\mathfrak{H}$ in the r.h.s. In the sequel we shall need the following

Lemma 4.1. *Let us consider the direct sum decomposition*

$$\mathfrak{g} = \mathfrak{H} \oplus \sum_\alpha \mathbb{R}(E_\alpha - E_{-\alpha}) \oplus \sum_\alpha \mathbb{R}\mathrm{i}(E_\alpha + E_{-\alpha}) \qquad (4.3)$$

(the sums ranging again over the positive roots) corresponding to (4.1). This decomposition is indeed orthogonal *with respect to the (negative) Killing form B (restricted to* $\mathfrak{g}$*).*

Proof. The lemma is easily proved via direct computation by observing that B is *symmetric* on $\mathfrak{g}_\mathbb{C}$ and that $B(E_\alpha, E_\beta) = 0$ for $\alpha + \beta \neq 0$. □

From now on let $\{t_k, r_\alpha, s_\alpha\}$ be an orthonormal basis in $\mathfrak{g}$ corresponding to the above decomposition, with

$$r_\alpha = (1/\sqrt{2})(E_\alpha - E_{-\alpha}), \qquad s_\alpha = (\mathrm{i}/\sqrt{2})(E_\alpha + E_{-\alpha}). \qquad (4.4)$$

It will be often convenient to switch to *hermitian* operators, adhering to the physical literature convention, and define

$$\tilde{r}_\alpha = \mathrm{i} r_\alpha, \quad \tilde{s}_\alpha = \mathrm{i} s_\alpha, \quad \tilde{t}_k = \mathrm{i} t_k \quad (\tilde{e}_i = \mathrm{i} e_i). \qquad (4.5)$$

We record the formula for the second-order *Casimir operator* C_2,

$$
\begin{aligned}
C_2 &= \sum_i \tilde{e}_i^2 = \sum_k \tilde{t}_k^2 + \sum_\alpha \tilde{r}_\alpha^2 + \sum_\alpha \tilde{s}_\alpha^2 \\
&= \sum_k \tilde{t}_k^2 + \sum_\alpha E_\alpha E_{-\alpha} + E_\alpha E_{-\alpha}. \qquad (4.6)
\end{aligned}
$$

Also notice the formula

$$2\langle E_\alpha\rangle\langle E_{-\alpha}\rangle=\langle\tilde{r}_\alpha^2\rangle+\langle\tilde{s}_\alpha^2\rangle\ . \tag{4.7}$$

A characteristic property of the positive Weyl chamber $\mathfrak{H}_+$ is that any adjoint orbit of G in $\mathfrak{g}$ meets $\mathfrak{H}_+$ in one and only one point, so that it may be used as a "moduli space" for the adjoint orbits [14,18].

Recall that the *moment map* in this framework reads (refs. [2,19,24], with different normalizations)

$$\mu([v])=-\mathrm{i}\,|v\rangle\langle v|\in \mathrm{u}(V)\ ,$$

namely

$$\begin{aligned}\mu(e_i)&:=(\mu([v]),e_i)=-\tfrac{1}{2}\mathrm{Tr}(-\mathrm{i}\,|v\rangle\langle v|e_i)\\ &=\tfrac{1}{2}\mathrm{Tr}(\mathrm{i}\,|v\rangle\langle v|(-\mathrm{i})\tilde{e}_i)\\ &=\tfrac{1}{2}\mathrm{Tr}(|v\rangle\langle v|\tilde{e}_i)=\tfrac{1}{2}\langle v|\tilde{e}_i v\rangle\ .\end{aligned}$$

In particular, we have

$$\|\mu\|^2([v])=\sum_i \mu(e_i)^2=\tfrac{1}{4}\sum_i \langle v|\tilde{e}_i v\rangle^2 \tag{4.8}$$

(we have used lemma 4.1). The action of H on V decomposes it into an orthogonal direct sum of *weight* spaces. We identify, as usual, a weight λ with its differential at the identity, namely, an element in $\mathfrak{H}^*\cong\mathfrak{H}$.

From a physical point of view, a state $[v]$ corresponding to a weight vector v is a *pure* state with respect to the Cartan subalgebra $\mathfrak{H}$: the observables of the form $\mathrm{i}h$, $h\in\mathfrak{H}$ form a maximal set of commuting observables. We have $\mathrm{i}hv=\lambda(\mathrm{i}h)v$ for a weight vector corresponding to λ, and we shall often write, in this case, $v=|\lambda\rangle$. Any vector state $[v]$ can be seen as a *mixture* of pure states: if $v=\sum_i c_i v_i$, with $\{v_i\}$ an orthonormal basis of V consisting of weight vectors, and $\|v\|=1$, then

$$[v]=\sum_i |c_i|^2[v_i],\quad \sum_i |c_i|^2=1$$

(upon restriction to $\mathfrak{H}$).

Any irreducible representation of G is described in terms of the so-called *highest weight* and according to the celebrated Borel–Weil–Bott theorem (see, e.g., ref. [5]), the representation space V can be viewed as the space of holomorphic sections of a suitable homogeneous holomorphic line on a natural Kähler manifold naturally associated with G (see also refs. [25] and [8]).

Let $|\lambda\rangle$ denote a unit *highest weight* (λ) vector, namely $E_\alpha|\lambda\rangle=0$ for any positive root α, and assume, for simplicity, that λ is *regular*, namely that the stabilizer group of $|\lambda\rangle$ coincides with $\mathfrak{H}$. (This is equivalent to the condition $\lambda\cdot\alpha\neq 0$. However, the whole discussion carries with minor modifications through the *singular* case.)

The G-orbit through $|\lambda\rangle$ is a *Kähler* submanifold of $P(V)$ diffeomorphic to

G/H (and denoted by O_λ) and called the *coherent state manifold*; this is easily checked to be consistent with the general scheme recalled in section 2. The *complex* structure of O_λ manifestly shows up upon recalling that

$$O_\lambda \cong G/H \cong G^{\mathrm{C}}/B^+ ,$$

where B^+ denotes the Borel subgroup of G^{C}, whose complex Lie algebra is generated by $\mathfrak{H}_{\mathrm{C}}$ and $\{E_\alpha\}$.

This can also be directly ascertained by using the explicit formulae describing the differential geometry of $P(V)$ discussed above, and observing that the tangent space to O_λ at $|\lambda\rangle$, say $T_{[|\lambda\rangle]}O_\lambda$, is generated by the fundamental vector fields at $[v]$ associated to the operators r_α and s_α above. On has

$$r^{\#}_\alpha|_{[v]} = |v\rangle\langle r_\alpha v| + |r_\alpha v\rangle\langle v| = -(1/\sqrt{2})(|v\rangle\langle E_{-\alpha}v| - |E_{-\alpha}v\rangle\langle v|) ,$$
$$s^{\#}_\alpha|_{[v]} = |v\rangle\langle s_\alpha v| + |s_\alpha v\rangle\langle v| = (1/\sqrt{2})(|v\rangle\langle \mathrm{i}E_{-\alpha}v| + |\mathrm{i}E_{-\alpha}v\rangle\langle v|) , \tag{4.9}$$

so

$$s^{\#}_\alpha|_{[v]} = -J|_{[v]} r^{\#}_\alpha|_{[v]} . \tag{4.10}$$

We are now in a position to state the following

Theorem 4.2. *Let O_λ be the Kähler G-orbit in $P(V)$ determined by the highest weight vector $|\lambda\rangle$ corresponding to the (non-singular) highest weight λ. Then, if α is a positive root:*

(i) the hamiltonians corresponding to r_α and s_α are canonically conjugate *at $|\lambda\rangle$;*

(ii) $$\Delta_\lambda(\tilde{r}_\alpha) = \Delta_\lambda(\tilde{s}_\alpha) = \tfrac{1}{2}|\langle\lambda|[\tilde{r}_\alpha, \tilde{s}_\alpha]\lambda\rangle| = \tfrac{1}{2}\lambda\cdot\alpha > 0 .$$

Proof.

(i) We have

$$\begin{aligned}\lambda^{\#}_{s\alpha}|_{[|\lambda\rangle]} &= \tfrac{1}{2}\mathrm{i}\langle s_\alpha\rangle^{\#}|_{[|\lambda\rangle]} = \tfrac{1}{2}\mathrm{i}(s^{\#}_\alpha|_{[|\lambda\rangle]}) \\ &= \tfrac{1}{2}\mathrm{i}(\mathrm{i}r^{\#}_\alpha|_{[|\lambda\rangle]}) = \tfrac{1}{2}\mathrm{i}J|_{[|\lambda\rangle]} r^{\#}_\alpha|_{[|\lambda\rangle]} \\ &= \tfrac{1}{2}\mathrm{i}J|_{[|\lambda\rangle]}\langle r_\alpha\rangle^{\#}|_{[|\lambda\rangle]} = J|_{[|\lambda\rangle]}\lambda^{\#}_{r\alpha}|_{[|\lambda\rangle]}\end{aligned}$$

[by (4.10) and theorem 2.4].

(ii) Follows from (i) and theorem 3.5, recalling that $[E_\alpha, E_{-\alpha}] = \mathrm{i}H_\alpha$. □

Remarks.

(i) As an easy application of the preceding theorem [part (i)] we recover in particular the well known formulae for SU(2), where r_α and s_α are the components J_x and J_y of the spin operator (see, e.g., ref. [26]). Notice that, given the result for SU(2), the general case is to be expected intuitively since any simple

Lie algebra is made up by suitably "pasting" together copies of su(2) [cf. (4.2)].

(ii) It is well known that the quantities $2(\lambda\cdot\alpha/\alpha\cdot\alpha)$, with α ranging over the *simple* (hence positive) roots, can be effectively used as representation labels, being the components of the highest weight with respect to a basis of simple roots (see, e.g., ref. [22]).

The preceding theorem provides a direct physical interpretation of these quantities, at least in the case of simply laced groups (where $\alpha\cdot\alpha$ is constant for all roots) in terms of minimum uncertainty of the corresponding canonically conjugate variables when they are measured in the highest weight state.

5. The projective space associated to an irreducible representation of a compact semisimple Lie group: the symplectic *G*-orbits

Let us now extend our discussion to the other symplectic *G*-orbits in $P(V)$. It is shown in ref. [21] (see also ref. [16]) that a *G*-orbit in $P(V)$ is symplectic (with the induced symplectic structure) if and only if it contains a weight vector $|\lambda\rangle$ such that $\lambda\cdot\alpha=0$ entails $E_{\pm\alpha}|\lambda\rangle=0$. In particular this orbit is Kähler if and only if $|\lambda\rangle$ is a highest weight vector. This can be also checked by means of the explicit formulae given above. The existence of symplectic, non-complex orbits has independently been noticed and exploited by Giavarini and Onofri in ref. [12] to understand the *migration* of spin in the context of degenerate Landau levels.

In view of theorem 2.1, the *symplectic* orbits provide *critical* orbits for $\|\mu\|^2$. This conclusion may also be reached through the remark of Ness [24] that weight vector orbits are always critical (see below). The converse to the preceding assertion does not hold. This will follow from the discussion below, but is also apparent from the combination of the Ness and Kostant–Sternberg theorems, since there are examples of weight vectors which do not give rise to symplectic orbits. Moreover, it will follow from the result described below that suitable *mixtures* of weight vector states may appear among the critical points of $\|\mu\|^2$.

We are now going to describe somewhat more explicitly the structure of the critical points of $\|\mu\|^2$ in $P(V)$. First of all, we recover, within our formalism, Ness' General Example [24]:

Proposition 5.1 [24]. *Weight vectors (and hence their G-orbits) provide critical points of* $\|\mu\|^2$.

Sketch of proof. We have to check that, for any $B\in \mathrm{u}(V)$,

$$B^{\#}(\|\mu\|^2)|_{[v]}=0\,,$$

and this is easily achieved upon recalling that $\langle v | r_\alpha v\rangle = \langle v | s_\alpha v\rangle = 0$. □

Now recall, from ref. [19], that the critical points of $\|\mu\|^2$ in a projective space are a union of sets on the form C_β, $\beta \in \mathfrak{H}_+$, with $C_\beta := G(Z_\beta \cap \mu^{-1}(\beta))$, and Z_β consists of the points $[v] \in P(V)$ such that, if $v = \sum_i c_i v_i$, $\sum_i |c_i|^2 = 1$, with respect to an orthonormal basis of V consisting of weight vectors v_i corresponding to weights λ_i, the only non-vanishing coefficients c_i are those for which $\lambda_i \cdot \beta = \|\beta\|^2$. The coefficients $|c_i|^2$ of the mixture $[v] = \sum_i |c_i|^2 [v_i]$ are determined by requiring β to be the closest point to the origin in $\mathfrak{H}$ lying in the positive Weyl chamber $\mathfrak{H}_+$. This description follows from the observation that the image of the moment map of a torus action on $P(V)$ is a convex polytope (its vertices being the weights of such an action). The corresponding result for a general symplectic manifold is a celebrated result of Atiyah [1], and Guillemin and Sternberg [14] (see also ref. [18] for further developments).

We shall also need the following lemma, which is established through an immediate calculation (via 2.11).

Lemma 5.2. *Let v be a (unit)* weight *vector in V. Let* $A \in \mathrm{u}(V)$. *Then*

$$g_{F[v]}(A^{\#}|_{[v]}, r_\alpha^{\#}|_{[v]}) = \mathrm{Re}\langle Av | r_\alpha v\rangle \;, \tag{5.1}$$

with a similar formula for s_α. □

Remark. Notice that (5.1) keeps trace only of the part of Av lying in the *orthogonal complement* V' in V of the line $[v]$ (as it should do, also in view of lemma 2.3). Also observe, for the sequel, that any vector in V', say δv, can be realized *either* as Av *or* as Bv, with some A (B) in $\mathrm{u}(V)$ $(\mathrm{iu}(V))$. This implies that

$$\delta v^{\#}|_{[v]} := |v + \delta v\rangle\langle v| + |v\rangle\langle v + \delta v|$$

is orthogonal to $r_\alpha^{\#}|_{[v]}$ if and only if

$$\mathrm{Re}\langle \delta v | \tilde{r}_\alpha v\rangle = 0 \,.$$

The following theorem specializes some general computations of Kirwan and Ness [19,24] to the particular setting of this section, and also provides a generalization of a result of Delbourgo [11,26].

Theorem 5.3.

(i) Let C_2 be the second-order Casimir introduced in section 4 and let

$$\Delta C_2 := \langle C_2\rangle - \sum_k \langle \tilde{t}_k\rangle^2 - \sum_\alpha \langle \tilde{r}_\alpha\rangle^2 - \langle \tilde{s}_\alpha\rangle^2$$

be the Delbourgo(–Perelomov) invariant dispersion. We have

$$\Delta C_2 = \langle C_2 \rangle - 4\|\mu\|^2 (\geqslant 0) , \tag{5.2}$$

whence ΔC_2 and $\|\mu\|^2$ have the same critical points.

(ii) The point $[v] \in Z_\beta \subseteq P(V)$, $\beta \in \mathfrak{H}_+$, is a critical point for $\|\mu\|^2$ if and only if, for any positive root α (and hence for all roots) — with respect to the above decomposition $v = \sum_i c_i v_i$, $\sum_i |c_i|^2 = 1$ — we have

$$\sum_{l,m} \bar{c}_l c_m \langle v_l | E_\alpha v_m \rangle = 0 . \tag{5.3}$$

(iii) Thus, a sufficient *condition for $[v]$ as above to be a critical point of $\|\mu\|^2$ is that in the decomposition $v = \sum_i c_i v_i$ no weight vectors are present which give rise to weights that differ by a root.*

(iv) The restriction of the Hessian of ΔC_2 at $[v]$ (with v a normalized weight vector corresponding to a weight λ) to the normal bundle with respect to the G-orbit through $[v]$ reads

$$H_{[v]}(\delta v, \delta v) = \tfrac{1}{2} \langle \delta v, \boldsymbol{H} \delta v \rangle = \tfrac{1}{2} g_{[v]}(\delta v, \boldsymbol{H} \delta v)$$

with $\boldsymbol{H} := \sum_h \lambda(t_h)(t_h - \lambda(t_h))$. From this, we get the following inequality [24]:

$$\mathrm{index}(H_{[v]}) \geqslant \#\{ \nu(\text{weight}) / \lambda \cdot \nu > \|\lambda\|^2, \nu \neq \lambda + \alpha, \alpha \in R \} .$$

Proof.

(i)

$$\Delta C_2 = \langle C_2 \rangle - \sum_k \langle \tilde{t}_k \rangle^2 - \sum_\alpha \langle \tilde{r}_\alpha \rangle^2 - \langle \tilde{s}_\alpha \rangle^2$$
$$= \langle C_2 \rangle - \sum_i \langle \tilde{e}_i \rangle^2 = \langle C_2 \rangle - 4\|\mu\|^2 ,$$

and the assertion follows since C_2 is a multiple of the identity due to the irreducibility of the representation.

(ii) Let $[v] \in Z_\beta$. We have

$$2\mu([v]) = \sum_k \langle v | \tilde{t}_k v \rangle \tilde{t}_k + \sum_\alpha \langle v | \tilde{r}_\alpha v \rangle \tilde{r}_\alpha + \sum_\alpha \langle v | \tilde{s}_\alpha v \rangle \tilde{s}_\alpha$$
$$= \sum_{k,l,m} \bar{c}_l c_m \langle v_l | \tilde{t}_k v_m \rangle \tilde{t}_k + \sum_{\alpha,l,m} \bar{c}_l c_m \langle v_l | \tilde{r}_\alpha v_m \rangle \tilde{r}_\alpha$$
$$+ \sum_{\alpha,l,m} \bar{c}_l c_m \langle v_l | \tilde{s}_\alpha v_m \rangle \tilde{s}_\alpha$$
$$= \sum_{l,m,k} \bar{c}_l c_m \langle v_l | \lambda_m(\tilde{t}_k) v_m \rangle \tilde{t}_k + \sum_{\alpha,l,m} \bar{c}_l c_m \langle v_l | \tilde{r}_\alpha v_m \rangle \tilde{r}_\alpha$$
$$+ \sum_{\alpha,l,m} \bar{c}_l c_m \langle v_l | \tilde{s}_\alpha v_m \rangle \tilde{s}_\alpha .$$

So, in order to have $2\mu([v]) = \beta \in \mathfrak{H}_+$ we must ensure that the coefficients of r_α

and s_α vanish identically, and this easily leads to (5.3) for any positive root. But, exploiting the fact that $E_\alpha^* = E_{-\alpha}$ we see that upon conjugating (5.3) with α positive, we get (5.3) for $-\alpha$ as well. So, if (5.3) holds, we have

$$2\mu([v]) = \sum_{k,l} |c_l|^2 \lambda_l(\tilde{t}_k)\tilde{t}_k = \sum_k \beta(\tilde{t}_k)\tilde{t}_k ,$$

i.e.,

$$2\mu([v]) = \sum_l [c_l|^2 \lambda_l = \beta ,$$

which allows one to determine the mixture coefficients $|c_l|^2$ in view of the above remarks. Each c_l remains undetermined up to a phase.

(iii) Follows immediately since $\langle v_l | E_\alpha v_m \rangle$ is zero whenever $\lambda_l \neq \lambda_m + \alpha$. As a retrospective check, we see that $2\mu([v]) = \beta$ *stabilizes* $[v]$, as it should do (since $\beta v = \|\beta\|^2 v$).

(iv) Follows by the straightforward computation of

$$\tfrac{1}{2}(\mathrm{d}^2/\mathrm{d}t^2)\|\mu\|^2([v(t)])|_{t=0} ,$$

where

$$[v(t)] := [v + t\delta v] = \|v + t\,\delta v\|^{-2} |v + t\,\delta v\rangle \langle v + t\,\delta v| ,$$

and in which we make use of the orthogonality (with respect to the metric g_{F}) condition to the tangent plane to the G-orbit in $[v]$, which reads (by lemmata 2.3, 4.1 and 5.2, and taking the remark preceding this theorem into due account)

$$\mathrm{Re}\langle \delta v | r_\alpha v \rangle = 0, \qquad \mathrm{Re}\langle \delta v | s_\alpha v \rangle = 0 .$$

We may obtain a more explicit formula as follows. Let $\{v_k\rangle$ be an orthonormal set of V' [with respect to $\langle \ | \ \rangle$] consisting of weight vectors pertaining to the representation weights λ_k such that $\lambda_k \neq \lambda + \alpha$, for any root α; then $\{v_k, v'_k := \mathrm{i} v_k\}$ provides an orthonormal set with respect to g_{F}, spanning a vector space W. Let $c_k = a_k + \mathrm{i} b_k$ and let $\delta v = \sum_k c_k v_k + \delta v' = \sum_k a_k v_k + b_k v'_k + \delta v'$, with $\delta v'$ orthogonal to W and $T_{[v]} O_\lambda$. We finally get

$$2H_{[v]}(\delta v, \delta v) = \sum_k \lambda \cdot (\lambda - \lambda_k)(a_k^2 + b_k^2) + \langle \delta v', \boldsymbol{H} \delta v' \rangle ,$$

whence, in particular, Ness' estimate [24] also follows. □

Observe that the minimal invariant uncertainty is realized precisely on the highest weight (λ) orbit (and we recover Delbourgo's theorem [11]); in fact in this case $4\|\mu\|^2$ attains its maximum $\|\lambda\|^2$, and it is also easy to see that $W = \{0\}$ (so the Hessian is positive definite).

We already observed that, if $v = v_l$ for a single weight λ_l in $\mathfrak{H}_+$, the orbit $G \cdot [v]$

is critical, and it is symplectic if the Kostant–Sternberg criterion [21,16] is satisfied. If λ_l is not in $\mathfrak{H}_+$, then it is Weyl-conjugate to a unique weight λ'_l in $\mathfrak{H}_+$ and the G-orbit passing through $[v'_l]$ (obvious notation) contains $[v_l]$; in fact, let $|v'\rangle = w|v\rangle$ [$w \in W$, the *Weyl group* associated to G)]; then, if $|v\rangle$ is a weight vector for λ, then $|v'\rangle$ is a weight vector for $\lambda \cdot w$, upon defining, for $t \in H$, $(\lambda \cdot w)(t) := \lambda(Ad_{w^{-1}}(t))$.

Remark. The non-minimal critical points are *unstable* in the sense of Mumford and of Guillemin and Sternberg, and live in the *null cone* of $P(V)$ [24,23].

Let us also finally notice the following

Corollary 5.4. *Let x and y (in $P(V)$) belong to different* symplectic *orbits of G in $P(V)$. They do* not *lie on the same orbit of G^{C}.*

Proof. x and y are critical points of $\|\mu\|^2$, by theorem 2.1, so the assertion follows from theorem 7.1 of ref. [24]. □

Final remarks. In this paper we focussed our attention on compact, connected, simple (or semisimple) Lie groups. We expect the basic picture outlined here to hold in the *non-compact semisimple* case as well. This question will possibly be addressed elsewhere.

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